Int. J. Solids Structures Vol. 25, No. 11, pp. 1279–1293, 1989
Printed in Great Britain.

0020-7683/89 $3.00+.00

ON INTERACTING CRACKS AND COMPLEX CRACK CONFIGURATIONS IN LINEAR ELASTIC MEDIA

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(*Received* 26 *May* 1988)

Abstract—A general and simple method is presented for the determination of stress intensity factors in elasticity problems involving several interacting cracks and complex crack shapes. The method uses a superposition scheme and an approximation of certain unknown crack-line tractions by a series of base functions. Crack interaction is accounted for by the stresses generated by an isolated line crack at a location of another crack when the former is subjected to the combination of base functions. The crack-line tractions are determined from the solution of a system of linear algebraic equations. Several examples which illustrate special forms of the method are presented. These include configurations like H-crack shapes motivated by studies in fracture of fibrous metal matrix composites. Comparison of results with available solutions shows that the method gives accurate results even when very few base functions are selected in the analysis.

1. INTRODUCTION

There are several situations in fracture mechanics which involve a complicated arrangement of cracks that is not amenable to a simple method of analysis. In some cases, the difficulty lies in having many cracks interacting with each other, e.g. when a single crack is embedded in a microcrack array. In other cases, as in crack branching phenomena, the complexity of the problem is due to the presence of irregular crack shapes consisting of several segments which form what is sometimes called a zig-zag or nonlinear crack. In relatively simple situations of multiple cracks, such as aligned cracks and crack branches, classical methods of analysis are applicable and they lead to elegant exact solutions, e.g. Erdogan (1962), Sih (1965). However, approximate methods are unavoidable in more complicated situations. Some existing studies have employed the representation of cracks by dislocations (Chudnovsky *et al.* 1987a,b; Vitek, 1977), which leads to integral equations which can be solved in an approximate way. Other investigations of zig-zag crack configurations use methods based on polynomial approximations and truncation of a conformal mapping function (Kitagawa and Yuuki, 1975, 1978). A particularly simple treatment of crack interaction phenomena has been introduced recently by Kachanov (1985, 1987) who showed that many multiple crack problems can be solved with the help of a superposition procedure which leads to a system of linear algebraic equations for certain equilibrated crack-line tractions. Another variant of a crack interaction method is given by Horii and Nemat-Nasser (1985) where inhomogeneity problems are also treated.

This paper presents a general and simple method for computation of stress fields and stress intensity factors in linear elastic media which contain several cracks arranged in a complicated configuration. The method uses a superposition technique which replaces a configuration of N cracks by means of N different problems, each involving an isolated crack located in an infinite medium and loaded by unknown tractions. Such representations were used by Collins (1962), Datsyshin and Savruk (1973) and more recently by Gross (1982), Chudnovsky and Kachanov (1983), Chudnovsky *et al.* (1987a,b), Horii and Nemat-Nasser (1983, 1985) and Chen (1984). In the present work a polynomial expansion for the unknown crack line tractions allows one to choose the number of suitable polynomials required for any desired accuracy. Once the approximating polynomials, called base functions in the sequel have been chosen, and the stress field due to a single crack in an infinite medium loaded by any such function is known, the problem reduces to the solution of a

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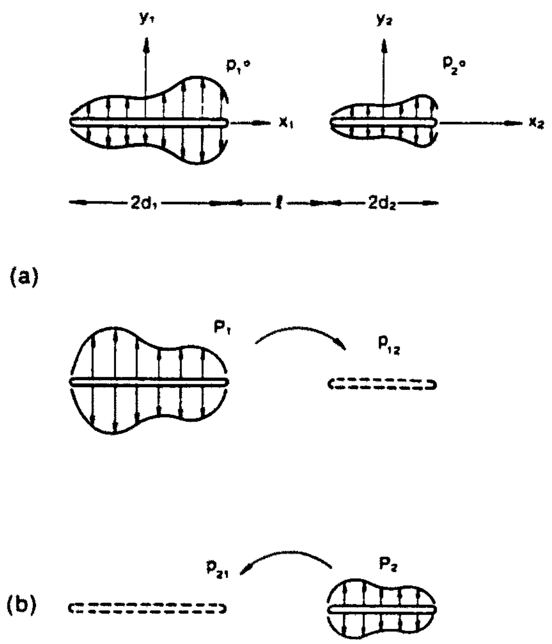

Fig. 1. Schematic of superposition of two aligned cracks in an infinite isotropic medium, loaded by normal tractions applied symmetrically at the faces of the cracks.

system of linear equations for certain unknown coefficients. We show that Kachanov's interaction scheme is a special case of the theory proposed herein, and that it corresponds to the case in which the unknown crack-line tractions are approximated only by their averages. In what follows, the method is formulated in the context of two-dimensional isotropic elasticity, but it can be readily extended to anisotropic media and applied, in principle, in three-dimensional problems.

The first section of the paper exposes the essence of the method by an example of its application to the simple case of two aligned cracks for which an exact solution actually exists (Erdogan, 1962). The second section describes the implementation of the method in the more complicated case of three cracks arranged in an H-crack configuration. The motivation for analysis of this problem will become evident in the companion papers in which the solution of the H-crack problem is employed in analysis of discrete plastic shear zones that are found at notches in fibrous metal matrix composites. In the last part of the paper, the method is used to evaluate stress intensity factors in the following crack configurations: (a) an H-crack loaded longitudinally, (b) an H-crack loaded transversely, (c) two parallel cracks under transverse normal stress, and (d) a T-crack loaded in tension. The results are compared with those which have been obtained in the literature with other approaches, such as dislocation distributions and conformal mapping techniques.

2. A SIMPLE EXAMPLE

The essential features of the method proposed herein can be illustrated by an example in which we consider two aligned cracks in an infinite isotropic medium, loaded by normal tractions applied symmetrically at the faces of the cracks (Fig. 1). Let the length of the cracks be $2d_\alpha$ ($\alpha = 1, 2$),† and let ℓ denote the distance that separates the adjacent crack tips. Define local coordinate systems (x_α, y_α) at the midpoints of the two cracks and apply normal stresses $p_\alpha^o(x_\alpha)$ which are symmetric with respect to the plane of the cracks. The tractions will be taken as negative when they open the crack and vice versa.

The two-crack problem of Fig. 1 can now be formulated as a superposition of two different problems in which each crack is regarded as a single crack in an infinite medium,

† The indices α or β denote quantities belonging to either crack, and assume values 1 or 2 in this section.

loaded by unknown surface tractions P_α which are to be determined (Fig. 1). In the first problem, let p_{12} be the stress at the imaginary location of crack 2, caused by an unknown traction P_1 applied at crack 1. In the second problem, let p_{21} be the stress at the imaginary location of crack 1, caused by an unknown traction P_2 applied at crack 2. For a superposition of these two problems to represent the solution of the original problem of Fig. 1, it is necessary to assure that

$$p_1^o(x_1) = P_1(x_1)+p_{21}(x_1) \tag{1a}$$

$$p_2^o(x_2) = P_2(x_2)+p_{12}(x_2). \tag{1b}$$

The unknown functions P_α will be estimated by a series of base functions which can be conveniently represented by Legendre polynomials

$$P_\alpha(x_\alpha) = \sum_{n=0}^{N} a_n^{(\alpha)}[-L_n^{(\alpha)}(\xi_\alpha)] \tag{2}$$

where $\xi_\alpha = (x_\alpha/d_\alpha)$, $a_n^{(\alpha)}$ are unknown coefficients and $L_n^{(\alpha)}$ are the Legendre polynomials defined by

$$L_0 = 1,\quad L_1(\xi) = \xi,\quad L_2(\xi) = \tfrac{3}{2}\xi^2 - \tfrac{1}{2}$$

$$L_3 = \tfrac{5}{2}\xi^3 - \tfrac{3}{2}\xi,\quad L_4(\xi) = \tfrac{35}{8}\xi^4 - \tfrac{30}{8}\xi^2 + \tfrac{3}{8}$$

$$L_n(\xi) = \frac{1}{2^n n!}\frac{d^n}{d\xi^n}(\xi^2-1)^n \quad n = 0, 1, 2, 3\ldots \tag{3}$$

In accordance with the adopted sign convention, the minus sign in (2) has been introduced to indicate that each base function loads the crack in an opening mode.

Define now by $f_{\beta\alpha}^{(n)}$ the stress resulting in the imaginary location of crack α when crack β is loaded by $-L_n^{(\beta)}(\xi_\beta)$. Since the functions L_n are polynomials, the influence functions $f_{\beta\alpha}^{(n)}(x_\alpha)$ can be generated without much difficulty by using the solution of a single crack located in an infinite medium and loaded by concentrated unit loads (see the Appendix). The functions $p_{\beta\alpha}$ are therefore given by

$$p_{\beta\alpha}(x_\alpha) = \sum_{n=0}^{N} a_n^{(\beta)} f_{\beta\alpha}^{(n)}(x_\alpha). \tag{4}$$

Finally, let the original loads on the crack surfaces be tractions resulting in an opening of the crack so that

$$p_\alpha^o(x_\alpha) = -\sigma_\alpha^o(x_\alpha),\quad \sigma_\alpha^o > 0. \tag{5}$$

Substitution of (2), (4) and (5) into (1) provides

$$\sum_{n=0}^{N} a_n^{(1)}L_n^{(1)}(x_1/d_1) = \sigma_1^o(x_1)+\sum_{n=0}^{N} a_n^{(2)}f_{21}^{(n)}(x_1) \tag{6a}$$

$$\sum_{n=0}^{N} a_n^{(2)}L_n^{(2)}(x_2/d_2) = \sigma_2^o(x_2)+\sum_{n=0}^{N} a_n^{(1)}f_{12}^{(n)}(x_2). \tag{6b}$$

Ideally, eqns (1) should be satisfied pointwise, but in what follows they are satisfied only approximately. If the summation in (6) is truncated at a finite N, $2N+2$ equations for the $2N+2$ unknowns a_n can be obtained in the following manner. Multiply (6a) by the Legendre polynomial $L_j(x_1/d_1)$, where $j = 0, \ldots, N$, and integrate from $-d_1$ to $+d_1$:

$$\sum_{n=0}^{N} a_n^{(1)} \int_{-d_1}^{+d_1} L_n(x_1/d_1) L_j(x_1/d_1)\, dx_1 = \int_{-d_1}^{+d_1} \sigma_1^o(x_1) L_j(x_1/d_1)\, dx_1 + \sum_{n=0}^{N} a_n^{(2)} \int_{-d_1}^{+d_1} L_j(x_1/d_1) f_{21}^{(n)}(x_1)\, dx_1. \quad (7)$$

Using the non-dimensional coordinate $\xi_1 = x_1/d_1$ and the orthogonality condition of the Legendre polynomials

$$\int_{-1}^{+1} L_j(\xi_1) L_i(\xi_1)\, d\xi_1 = \delta_{ij}[2/(2i+1)], \quad (8)$$

provides $N+1$ equations

$$d_1[2a_j^{(1)}/(2j+1)] = S_{1j}^o + \sum_{n=0}^{N} F_{21}^{(n,j)} a_n^{(2)} \quad (9)$$

where

$$S_{1j}^o = \int_{-d_1}^{+d_1} \sigma_1^o(x_1) L_j(x_1/d_1)\, dx_1, \quad F_{21}^{(n,j)} = \int_{-d_1}^{+d_1} f_{21}^{(n)}(x_1) L_j(x_1/d_1)\, dx_1. \quad (10)$$

A similar procedure applied to equation (6b) yields the additional $N+1$ equations

$$d_2[2a_j^{(2)}/(2j+1)] = S_{2j}^o + \sum_{n=0}^{N} F_{12}^{(n,j)} a_n^{(1)} \quad (11)$$

where

$$S_{2j}^o = \int_{-d_2}^{+d_2} \sigma_2^o(x_2) L_j(x_2/d_2)\, dx_2, \quad F_{12}^{(n,j)} = \int_{-d_2}^{+d_2} f_{12}^{(n)}(x_2) L_j(x_2/d_2)\, dx_2. \quad (12)$$

Equations (9) and (11) can now be solved for the $2N+2$ unknowns $a_n^{(\alpha)}$. Then, stress intensity factors at the tips of the two cracks and the stress field can be obtained by reference to the original superposition scheme of Fig. 1 and eqn (1).

The stress intensity factors at the four crack tips follow by integration of the well known expression which gives this factor for a single crack loaded by a concentrated unit load. For example, for crack 1 one obtains

$$K_1^{\pm} = \frac{1}{\sqrt{\pi d_1}} \int_{-d_1}^{+d_1} \sqrt{\frac{d_1 \pm t_1}{d_1 \mp t_1}}\, \sigma(t_1)\, dt_1 \quad (13)$$

where $\sigma(t_1)$, as explained below, is given by the right-hand side of (6a), and the + and − signs correspond to the values of K_1 at the right and left tips, respectively.

The stresses induced by the two loaded cracks at any point in the elastic medium can be found, in principle, as a sum of the stress induced by crack 1 loaded by $P_1(x_1)$ plus that due to crack 2 loaded by $P_2(x_2)$. However, since eqns (6) are not satisfied pointwise, the option arises whether the left- or the right-hand side of (6) should be used to represent $P_\alpha(x_\alpha)$. To clarify this question, suppose that only one term is used ($N = 0$) in the series (6). In this particular case, the right-hand side of (6) describes loading of the crack by the originally prescribed tractions and by the nonuniform stress which is induced by the presence of the other loaded crack, while the left-hand side of (6) represents loading of the cracks by constant stresses which are the averages of the two preceding terms at each crack. Under

Table 1.

	$K_I/p^o\sqrt{\pi d}$ at inner tip			$K_I/p^o\sqrt{\pi d}$ at outer tip		
$\delta = (\ell/2)/(\ell/2)+2d$	Exact results Erdogan (1962)	Kachanov's predictions Kachanov (1985)	Present predictions	Exact results Erdogan (1962)	Kachanov's predictions Kachanov (1985)	Present predictions
0.01	2.371	2.138	2.390	1.184	1.175	1.186
0.001	5.395	3.401	5.534	1.244	1.214	1.261
0.0001	13.347	4.731	11.646	1.280	1.227	1.325
10^{-6}	93.03	7.309	26.320	1.321	1.233	1.379

such circumstances, loading of the cracks by the right-hand side terms of (6) appears to be preferable both in determination of the local stresses, and in evaluation of the stress intensity factors in (13).

The results predicted by the above analysis for the case of equal length cracks ($d_1 = d_2 = d$) loaded by a uniform load p^o are illustrated in Table 1, where a comparison is also presented with the exact solution Erdogan (1962) and the predictions of the Kachanov's method. Our results were obtained with five approximating polynomials, and as mentioned above, Kachanov's results correspond to the case of $N = 0$. The cracks were chosen very close to each other to allow a strong interaction. At a value of $\delta = 0.01$, the present predictions are excellent. Even for $\delta = 0.001$, the error at the inner crack tip does not exceed 2.6%. As expected, the accuracy decreases when the cracks approach each other further, especially for the stress intensity factor at the inner tip. It is interesting to note however that when the cracks almost touch each other ($\delta = 10^{-6}$) the error for the stress intensity factor at the outer tip is only of the order of 4%.

In conclusion of this simple exposition of the proposed superposition method, we note that it is not necessary to use only the Legendre polynomials to approximate the unknown tractions P_α. Indeed, in the examples which follow we show that symmetric crack configurations, such as the H- and T-cracks, suggest the use of other base functions which can be incorporated in the solution. It is finally noted that crack closure effects have been disregarded in the present paper.

3. ANALYSIS OF THE METHOD

The procedure outlined above can be readily generalized to the case of many interacting cracks which are distributed in a specified manner in a plane, or in a three-dimensional solid. We limit our treatment to the plane case. In particular, consider M interacting cracks in the xy-plane. The cracks are no longer aligned in any special way, and each crack can assume any given orientation with respect to the coordinate axes. Let d_α denote the half-length of crack α, and let $x_\alpha y_\alpha$ denote the local coordinate system of each crack positioned at the mid point of each crack with $y_\alpha = 0$ denoting the crack plane. The Greek letters α and β will be used to denote individual cracks such that $\alpha = 1, 2, \ldots, M$, and $\beta = 1, 2, \ldots, M$, $\alpha \neq \beta$. Each crack is loaded by prescribed tractions ; in the local coordinate system of each crack, $p^o_\alpha(x_\alpha)$ will denote the normal component and $s^o_\alpha(x_\alpha)$ the shear component of the local traction. As before, the tractions $p^o_\alpha(x_\alpha)$ which result in an opening mode will be taken as negative. The $s^o_\alpha(x_\alpha)$ tractions will be negative when the tractions at the upper face ($y = 0^+$) of the crack point in the x_α-direction.

The solution of the many crack problem can again be found by the superposition of N different problems in which each crack is regarded as a single crack located in an infinite medium and loaded by unknown tractions $P_\alpha(x_\alpha)$ and $S_\alpha(x_\alpha)$ to be determined. Let $p_{\beta\alpha}(x_\alpha)$ and $s_{\beta\alpha}(x_\alpha)$ denote the normal and shear tractions respectively which are caused at the imaginary location of crack α by tractions $P_\beta(x_\beta)$ and $S_\beta(x_\beta)$ applied at the surrounding cracks, β, $\alpha \neq \beta$. Therefore in analogy to (1), the prescribed tractions at each crack are expressed as follows:

$$p_\alpha^o(x_\alpha) = P_\alpha(x_\alpha) + \sum_{\beta=1}^{M} p_{\beta\alpha}(x_\alpha)$$

$$s_\alpha^o(x_\alpha) = S_\alpha(x_\alpha) + \sum_{\beta=1}^{M} s_{\beta\alpha}(x_\alpha). \quad \alpha \neq \beta. \tag{14}$$

The tractions $P_\alpha(x_\alpha)$, $S_\alpha(x_\alpha)$ are expanded in the local coordinate systems x_α in a series of base functions which we chose again to be the Legendre polynomials (3):

$$P_\alpha(x_\alpha) = \sum_{n=0}^{N} a_n^{(\alpha)}[-L_n^{(\alpha)}(\xi_\alpha)]$$

$$S_\alpha(x_\alpha) = \sum_{n=0}^{N} b_n^{(\alpha)}[-L_n^{(\alpha)}(\xi_\alpha)] \tag{15}$$

where $\xi_\alpha = x_\alpha/d_\alpha$. Other suitable base functions may be chosen as needed.

Next, influence functions are defined such that they describe the tractions at the imaginary location of the crack α, caused by the base functions $-L_n^{(\beta)}(\xi_\beta)$ applied at cracks β. Four such functions are needed. The function $f_{\beta\alpha}^{(n)}$ represents the normal stress on the imaginary location of crack α due to the presence of a solitary crack β which is loaded normally by a base function of order n. In this spirit, we state schematically the following definitions:

$f_{\beta\alpha}^{(n)}(x_\alpha)$ is normal stress induced at location α by a normal stress applied at crack β;
$g_{\beta\alpha}^{(n)}(x_\alpha)$ is shear stress induced at location α by a normal stress applied at crack β;
$h_{\beta\alpha}^{(n)}(x_\alpha)$ is normal stress induced at location α by a shear stress applied at crack β;
$q_{\beta\alpha}^{(n)}(x_\alpha)$ is shear stress induced at location α by a shear stress applied at crack β.

These definitions lead to equations which are analogous to (4):

$$p_{\beta\alpha}(x_\alpha) = \sum_{n=0}^{N} (a_n^{(\beta)} f_{\beta\alpha}^{(n)}(x_\alpha) + b_n^{(\beta)} h_{\beta\alpha}^{(n)}(x_\alpha))$$

$$s_{\beta\alpha}(x_\alpha) = \sum_{n=0}^{N} (a_n^{(\beta)} g_{\beta\alpha}^{(n)}(x_\alpha) + b_n^{(\beta)} q_{\beta\alpha}^{(n)}(x_\alpha)). \tag{16}$$

Using (15) and (16) one can rewrite (14) in the form:

$$p_\alpha^o(x_\alpha) = \sum_{n=0}^{N} a_n^{(\alpha)}[-L_n^{(\alpha)}(\xi_\alpha)] + \sum_{\beta=1}^{M}\sum_{n=0}^{N} [a_n^{(\beta)} f_{\beta\alpha}^{(n)}(x_\alpha) + b_n^{(\beta)} h_{\beta\alpha}^{(n)}(x_\alpha)]$$

$$s_\alpha^o(x_\alpha) = \sum_{n=0}^{N} b_n^{(\alpha)}[-L_n^{(\alpha)}(\xi_\alpha)] + \sum_{\beta=1}^{M}\sum_{n=0}^{N} [a_n^{(\beta)} g_{\beta\alpha}^{(n)}(x_\alpha) + b_n^{(\beta)} q_{\beta\alpha}^{(n)}(x_\alpha)] \tag{17}$$

with $\beta \neq \alpha$.

Now, each equation is multiplied by $L_j(\xi_\alpha)$, $j = 0, 1, 2, \ldots, N$, and integrated with respect to x_α in the interval from $-d_\alpha$ to $+d_\alpha$. With the help of the orthogonality condition (8), which can be easily adjusted to the indicated interval, one finds

$$p_\alpha^{(j)} = -a_j^{(\alpha)}[2d_\alpha/(2j+1)] + \sum_{\beta=1}^{M}\sum_{n=0}^{N} [a_n^{(\beta)} F_{\beta\alpha}^{(n,j)} + b_n^{(\beta)} H_{\beta\alpha}^{(n,j)}] \tag{18}$$

$$s_\alpha^{(j)} = -b_j^{(\alpha)}[2d_\alpha/(2j+1)] + \sum_{\beta=1}^{M}\sum_{n=0}^{N} [a_n^{(\beta)} G_{\beta\alpha}^{(n,j)} + b_n^{(\beta)} H_{\beta\alpha}^{(n,j)}] \tag{19}$$

where the following definitions have been introduced:

$$\begin{Bmatrix} p_\alpha^{(j)} \\ s_\alpha^{(j)} \end{Bmatrix} = \int_{-d_\alpha}^{+d_\alpha} \begin{Bmatrix} p_\alpha^o(x_\alpha) \\ s_\alpha^o(x_\alpha) \end{Bmatrix} L_j^{(\alpha)}(x_\alpha/d_\alpha)\,\mathrm{d}x_\alpha \tag{20}$$

and the formula

$$Z_{\beta\alpha}^{(n,j)} = \int_{-d_\alpha}^{+d_\alpha} z_{\beta\alpha}^{(n)}(x_\alpha) L_j^{(\alpha)}(x_\alpha/d_\alpha)\,\mathrm{d}x_\alpha \tag{21}$$

where $Z_{\beta\alpha}^{(n,j)}$ assumes, in turn, the values of $F_{\beta\alpha}^{(n,j)}$, $H_{\beta\alpha}^{(n,j)}$, $G_{\beta\alpha}^{(n,j)}$ and $Q_{\beta\alpha}^{(n,j)}$, while $z_{\beta\alpha}^{(n)}$ assumes, in turn, the values of $f_{\beta\alpha}^{(n)}$, $h_{\beta\alpha}^{(n)}$, $g_{\beta\alpha}^{(n)}$ and $q_{\beta\alpha}^{(n)}$.

We now recall that $\alpha = 1, 2, \ldots, M$; $\beta = 1, 2, \ldots, M$; $j = 0, 1, 2, \ldots, N$; and $n = 0, 1, 2, \ldots, N$. Therefore (18) and (19) are the required $2M(N+1)$ equations for the $M(N+1)$ unknowns $a_j^{(\alpha)}$, and the $M(N+1)$ unknowns $b_j^{(\alpha)}$, which are needed to find the tractions $p_{\beta\alpha}$ and $s_{\beta\alpha}$ in (16).

As pointed out at the end of Section 2, the stress intensity factors at each crack α and the stresses in the vicinity of the crack should be determined by regarding each crack as a solitary crack in an infinite medium loaded by surface tractions given, in analogy with (6), by the differences between the terms $p_\alpha^o(x_\alpha)$, or $s_\alpha^o(x_\alpha)$, and the second terms on the right-hand sides of the respective eqns (17).

4. THE H-CRACK

The results of the previous section are implemented here to analyze an H-crack configuration consisting of three mutually perpendicular cracks (Fig. 2). In this implementation the single H-crack is represented by three line cracks which touch each other. This representation will result in the usual stress singularities at the points at which the three cracks touch each other, no matter what is the distance of separation and thus would give unrealistic results at that location. However, the change in the actual geometry of the problem makes it possible to use the present method for evaluation of the stress intensity factors at the tips of cracks 2 and 3. These are of interest in applications of the present method to certain fracture problems in fibrous metal matrix composites. Comparisons with other solutions shown in the sequel indicate that the method gives good approximations of these stress intensity factors.

In Fig. 2 the middle crack of length $2L$ is denoted by the index 1, and the right and left cracks of length $2R$ by indices 2 and 3, respectively. Local coordinates (x_α, y_α) are defined at the center of each crack. The solution will be sought for certain symmetric overall loads which can be reduced to the normal and shear tractions indicated in Fig. 2b. The normal and shear tractions are denoted by p_α^o, s_α^o, where α (or β) $= 1, 2, 3$, and are applied symmetrically to both faces of each crack. They satisfy the following conditions:

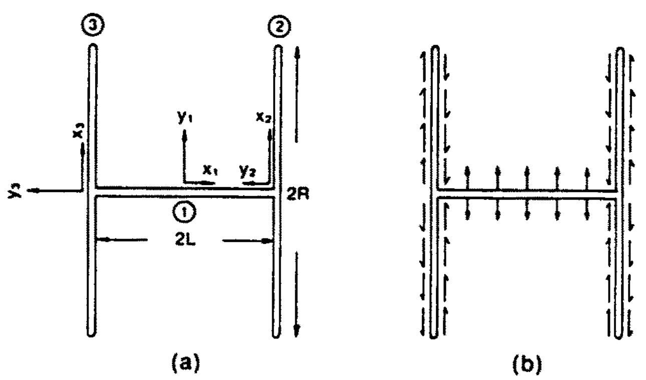

Fig. 2. Symmetry of geometry and loads in an H-crack type configuration.

$$p^o_\alpha(x_\alpha) = p^o_\alpha(-x_\alpha), \quad \alpha = 1,2,3; \quad p^o_2(x_2) = p^o_3(x_3) \text{ at } x_2 = x_3;$$
$$s^o_1(x_1) = 0; \quad s^o_\alpha(x_\alpha) = -s^o_\alpha(-x_\alpha); \quad \alpha = 2,3;$$
$$s^o_2(x_2) = -s^o_3(x_3) \text{ at } x_2 = x_3. \tag{22}$$

With reference to (15), we now select the representation of the functions $P_\alpha(x_\alpha)$ and $S_\alpha(x_\alpha)$. The forms that reflect the symmetry conditions (22) are

$$P_\alpha(x_\alpha) = \sum_{n=0,2,4,\ldots} a^{(\alpha)}_n[-L^{(\alpha)}_n(\xi_\alpha)] \quad \alpha = 1,2,3 \tag{23a}$$

$$S_2(x_2) = \sum_{n=-1,1,3,\ldots} b^{(2)}_n[-L^{(2)}_n(\xi_2)] \tag{23b}$$

$$S_3(x_3) = \sum_{n=-1,1,3,\ldots} b^{(3)}_n[-L^{(3)}_n(\xi_3)] \tag{23c}$$

where $\xi_1 = x_1/L$, $\xi_2 = x_2/R$, $\xi_3 = x_3/R$. For $n \geqslant 0$, the functions $L_n(\xi_\alpha)$ are the Legendre polynomials (3), but for $n = -1$, $\alpha = 2,3$, we choose the function

$$L^{(\alpha)}_{-1}(\xi_\alpha) = \begin{cases} +1 & \text{for } 0 \leqslant \xi_\alpha \leqslant 1 \\ -1 & \text{for } -1 \leqslant \xi_\alpha \leqslant 0 \end{cases} \quad \alpha = 2,3 \tag{24}$$

which, of course, is not a Legendre polynomial. It has been introduced to allow for the possibly discontinuous shear stress distribution at the mid-points of the side cracks. We also note that due to the assumed symmetry of the loads (22), the P_α is symmetric in ξ_α, and thus was represented only by Legendre polynomials of even order; S_α, on the other hand, is anti-symmetric with respect to ξ_α, $\alpha = 2,3$. Furthermore, the absence of external shear loads on crack 1, and the presence of symmetric loads on cracks 2 and 3 induce no shear stresses on crack 1, which implies that $S_1 = 0$. Note also that the base functions on cracks 2 and 3 have been chosen as mirror images of each other. That, together with the symmetric external loads implies that:

$$a^{(2)}_n = a^{(3)}_n \quad n = 0,2,4,\ldots$$
$$b^{(2)}_n = b^{(3)}_n \quad n = -1,1,3,\ldots. \tag{25}$$

The next step in the solution is the evaluation of the influence functions which is outlined in detail in the Appendix. With reference to (16) and (23) we have for crack 1

$$p_{\beta 1} = \sum_{n=0,2,4,\ldots} a^{(\beta)}_n f^{(n)}_{\beta 1} + \sum_{n=-1,1,3} b^{(\beta)}_n h^{(n)}_{\beta 1}, \quad \beta = 2,3. \tag{26}$$

Substitution of (26) and (23a) into (17a), with $\alpha = 1$, gives

$$p^o_1(x_1) = \sum_{n=0,2,4,\ldots} a^{(1)}_n(-L^{(1)}_n) + \sum_{\beta=2,3}\sum_{n=0,2,4,\ldots} a^{(\beta)}_n f^{(n)}_{\beta 1} + \sum_{\beta=2,3}\sum_{n=-1,1,3,\ldots} b^{(\beta)}_n h^{(n)}_{\beta 1}. \tag{27}$$

By the prescribed symmetry, all the functions in (27) are even in x_2. The last equation will now be multiplied by $L^{(1)}_j$, $j = 0,2,4,\ldots$, and integrated from 0 to L. That gives the specific form of equation (18):

$$p^{(j)}_1(x_1) = -\frac{\delta_{ij}L}{2i+1}a^{(1)}_i + \sum_{\beta=2,3}\sum_{n=0,2,4,\ldots} a^{(\beta)}_n F^{(n,j)}_{\beta 1} + \sum_{\beta=2,3}\sum_{n=-1,1,3,\ldots} b^{(\beta)}_n H^{(n,j)}_{\beta 1} \tag{28}$$

with $j = 0,2,4,\ldots$ and the coefficients being given by

$$p_1^{(j)} = \int_0^L p_1^o(x_1)[L_j^{(1)}(x_1/L)]\,dx_1$$

$$F_{\beta 1}^{(n,j)} = \int_0^L f_{\beta 1}^{(n)}(x_1)[L_j^{(1)}(x_1/L)]\,dx_1, \quad H_{\beta 1}^{(n,j)} = \int_0^L h_{\beta 1}^{(n)}(x_1)[L_j^{(1)}(x_1/L)]\,dx_1. \tag{29}$$

Note here that all Legendre polynomials in (27) are even in x_1, hence their orthogonality property is implemented by integration of (8) from 0 to L.

We now turn our attention to crack 2, and consider first the normal stresses. A treatment similar to that leading to (18) or (28) gives

$$p_2^{(j)}(x_2) = -\frac{\delta_{ij}R}{2i+1}a_i^{(2)} + \sum_{\beta=1,3}\sum_{n=0,2,4,\ldots} a_n^{(\beta)}F_{\beta 2}^{(n,j)} + \sum_{n=-1,1,3,\ldots} b_n^{(3)}H_{32}^{(n,j)} \tag{30}$$

with $j = 0, 2, 4, \ldots$, and with coefficients $p_2^{(j)}$, $F_{\beta 2}^{(n,j)}$, $H_{32}^{(n,j)}$ given again by (20) and (21) where the integration is this time from 0 to R and $\alpha = 2$.

The shear stresses on crack 2 follow from the appropriate form of (17) which in the present case becomes

$$s_2^o(x_2) = \sum_{n=-1,1,3,\ldots} b_n^{(2)}(-L_n^{(2)}) + \sum_{\beta=1,3}\sum_{n=0,2,4,\ldots} a_n^{(\beta)}g_{\beta 2}^{(n)} + \sum_{n=-1,1,3,\ldots} b_n^{(3)}q_{32}^{(n)}. \tag{31}$$

At this time however, all the functions appearing in (31) are anti-symmetric in x_2, and $L_{-1}^{(2)}$ is not a Legendre polynomial. But the procedure remains much the same, with the following differences. Equation (31) is multiplied by $L_j^{(2)} = -1, 1, 3, \ldots$, respectively, and integrated with respect to x_2 from 0 to R:

$$s_2^{(j)} = q_j + \sum_{\beta=1,3}\sum_{n=0,2,4,\ldots} a_n^{(\beta)}G_{\beta 2}^{(n,j)} + \sum_{n=-1,1,3,\ldots} b_n^{(3)}Q_{32}^{(n,j)} \tag{32}$$

with $j = -1, 1, 3, \ldots$, and

$$s_2^{(j)} = \int_0^R s_2^o(x_2)[L_j^{(2)}(x_2/R)]\,dx_2, \quad j = -1, 1, 3, \ldots \tag{33}$$

$$q_{-1} = -b_{-1}R - \sum_{n=1,3,\ldots}\int_0^R b_n^{(2)}L_n^{(2)}(x_2/R)L_{-1}^{(2)}(x_2/R)\,dx_2 \tag{34a}$$

where, for $x_2 \geqslant 0$, $L_{-1}^{(2)}(x_2/R) = 1$; for $j = 1, 3, \ldots$, the expression for q_j is

$$q_j = -b_{-1}^{(2)}\int_0^R -L_{-1}^{(2)}(x_2/R)L_j^{(2)}(x_2/R)\,dx_2 - \delta_{ij}[R/(2i+1)]b_i^{(2)}. \tag{34b}$$

Finally,

$$G_{\beta 2}^{(n,j)} = \int_0^R g_{\beta 2}^{(n)}(x_2)[L_j^{(2)}(x_2/R)]\,dx_2, \quad Q_{32}^{(n,j)} = \int_0^R q_{32}^{(n)}(x_2)[L_j^{(2)}(x_2/R)]\,dx_2. \tag{35}$$

From the invoked symmetry conditions of the problem, and in view of (25), the fulfillment of (28), (30), and (32) ensures that the equilibrium conditions on crack 3 are automatically satisfied. It is finally noted that the series (23a) can be truncated at $n = N$ for $\alpha = 1$, and for $n = M \neq N$ for $\alpha = 2, 3$, depending on the geometry and the boundary conditions. In the actual computations described in the sequel, we choose $n = 0, 2, 4$ for

$a_n^{(1)}$; $n = 0, 2$ for $a_n^{(2)}$; and $n = -1, 1, 3$ for $b_n^{(2)}$; hence the solution of the H-crack problem was found from only eight linear algebraic equations for the eight unknowns $a_n^{(\alpha)}$, $b_n^{(\alpha)}$.

Once the unknown coefficients have been determined, the stress intensity factors at the tips of the H-crack can be found in analogy to (13). For the tractions on the faces of crack 2, for example,

$$\begin{aligned} p_2^o(x_2) &= -\sigma_2^o(x_2), \quad \sigma^o > 0 \\ s_2^o(x_2) &= -\tau_2^o(x_2), \quad \tau^o > 0. \end{aligned} \tag{36}$$

The stress intensity factors K_I and K_{II} at the upper crack tip in Fig. 2 are given by

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{1}{\sqrt{\pi R}} \int_{-R}^{+R} \sqrt{\frac{R+t}{R-t}} \begin{Bmatrix} \sigma(t) \\ \tau(t) \end{Bmatrix} \mathrm{d}t \tag{37}$$

with

$$\begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} = \begin{Bmatrix} \sigma_2^o(x_2) \\ \tau_2^o(x_2) \end{Bmatrix} + \begin{Bmatrix} p_{12}(x_2) + p_{32}(x_2) \\ s_{12}(x_2) + s_{32}(x_2) \end{Bmatrix}. \tag{38}$$

5. NUMERICAL EXAMPLES

We now proceed to present the stress intensity factors for the H-crack and for several other crack configurations which were obtained from the method proposed herein. Comparisons with other solutions will be also shown.

5.1. *An H-crack under uniform axial normal stress*

Figure 3 illustrates the crack configuration and the loading conditions, and present the stress intensity factor results. These are the K_I and K_{II} factors at the upper tip A of crack 2. The figure also shows a comparison with the results found by Vitek (1977) who modeled the cracks by a distribution of dislocations. The stress intensity factors are given for ratios of R/L from 0 to 10; such ratios are often found in the composite fracture problems mentioned earlier. This contrasts with most other solutions in the literature which are concerned with crack branching phenomena that involve only small ratios of R/L. Vitek's solution is exceptional in that regard, and thus provides an opportunity for comparisons

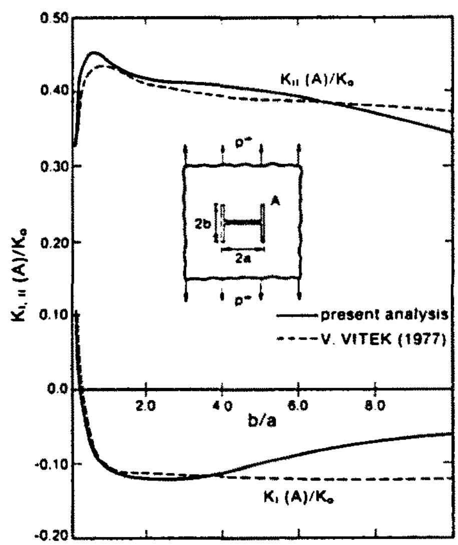

Fig. 3. Comparison of K_I and K_{II} for an H-crack under uniform normal stress at infinity.

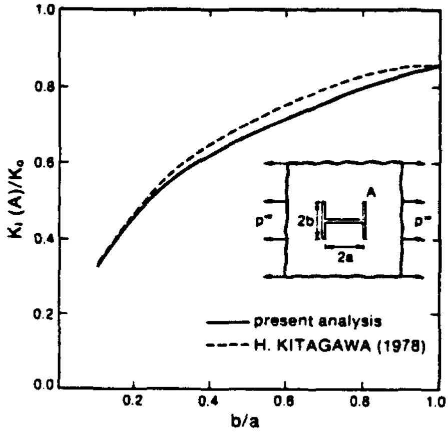

Fig. 4. Comparison of K_I for an H-crack under uniform transverse normal stress at infinity.

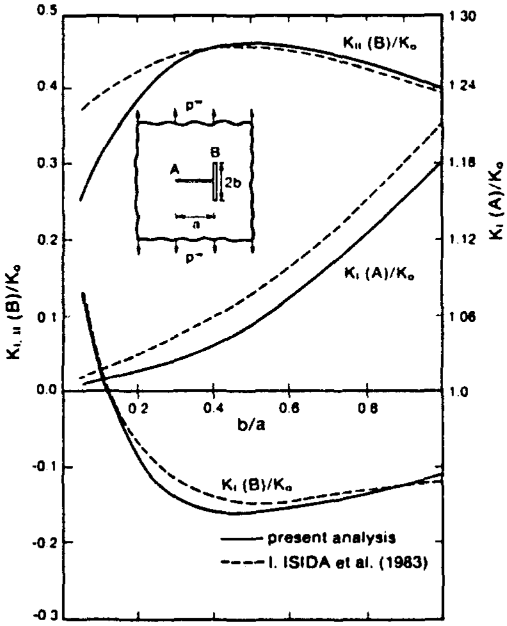

Fig. 5. Comparison of K_I and K_{II} for a T-crack under uniform normal stress at infinity.

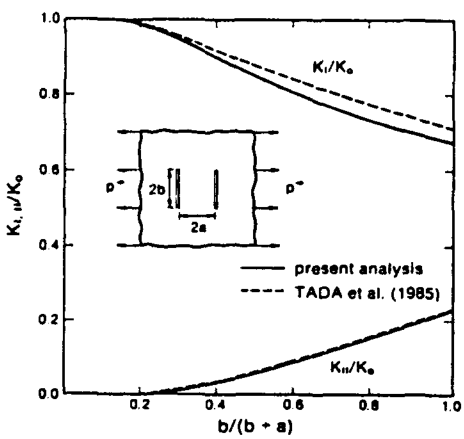

Fig. 6. Comparison of K_I and K_{II} for two parallel cracks under uniform transverse normal stress at infinity.

in the wider range of R/L. Both in this case and in those which follow, the series in (23a) was truncated at $n = 4$ for $\alpha = 1$, at $n = 2$ for $\alpha = 2, 3$, and $n = 3$ was used for (23b,c). That resulted in eight simultaneous equations for eight unknowns. It is noted that the success of the method is more pronounced in predicting the dominant stress intensity factor K_{II}. It is presumed however that a higher number of polynomials would have resulted in better accuracy. A negative value in K_I is of course to be interpreted as the closure of that crack for the corresponding loads, a phenomenon which as mentioned above was not accounted for in the present study. These negative values however are useful when using a superposition procedure for a different set of loads.

5.2. *An H-crack under uniform transverse normal stress*

Figure 4 indicates this loading configuration and exhibits the predicted results for the dominant stress intensity factor K_I at the tip A of crack 2. A comparison is shown with the results obtained by Kitagawa and Yuuki (1978) which used a conformal mapping technique (see also the handbook by Murakami, 1987, p. 389). The stress intensity factor K_{II} which was reported in this last reference to be negligible and was not matched accurately by our predictions ($|K_{II}/p^{\infty}\sqrt{\pi a}| \leqslant 0.1$). Given the small value of this quantity however, no effort was made to improve the agreement by increasing the number of base functions.

5.3. *Two parallel cracks under uniform transverse normal stress*

This is a special case of the H-crack configuration in which the middle crack has been eliminated. Figure 5 shows the results, and also a comparison with the values reported in Isida (1972), which were found using the series expansion of complex potential technique (see also the handbook by Tada *et al.*, 1985, pp. 14–17).

5.4. *A T-crack configuration*

Figure 6 shows the crack configuration and loading condition and presents the predicted results comparing them with those given by Kitagawa and Yuuki (1975). It should be mentioned here that since the T-crack has only one plane of symmetry the implementation of the method required some changes in the procedure described in Section 4. The approximating polynomials for the horizontal crack now were not $(L_0^{(1)}, L_2^{(1)}, L_4^{(1)})$ but $(L_0^{(1)}, L_1^{(1)}, L_2^{(1)})$.

CONCLUSION

The outstanding advantage of the method presented in this paper is its simplicity. Once the stress fields generated by a solitary crack under power type tractions are found, the implementation of the method necessitates only the solution of a set of linear algebraic equations for the unknown coefficients.

The presented examples show that the method predicts accurately stress intensity factors in multiple crack problems even when the cracks are very close to each other. Complex crack patterns, such as the H- and T-cracks are dealt with successfully by coalescing line cracks into desired configurations. As expected, the stresses at the points of coalescence are not well described, but the stress intensity factors at the tips of interacting cracks are predicted with remarkable accuracy even when the tips are in close proximity.

Acknowledgements—The first author is indebted to Professor G. J. Dvorak for the visiting appointment at the Civil Engineering Department at RPI during 1987/1988. Helpful discussions with Professor Y. A. Bahei-El-Din are gratefully acknowledged. The authors are grateful for financial support of this work by the Mechanics Division of the Office of Naval Research. Dr Yapa Rajapakse served as program monitor.

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APPENDIX

In this appendix analytical expressions will be derived for the stress fields generated by a single crack located in an infinite isotropic medium and subjected to certain loading distributions. Specifically, the results of this appendix will be used to generate the influence functions $f_{\beta\alpha}^{(m)}$, $g_{\beta\alpha}^{(m)}$, $h_{\beta\alpha}^{(m)}$, $q_{\beta\alpha}^{(m)}$.

Consider a single crack of length 2ℓ located in an isotropic medium. Let the crack be subjected to concentrated normal and shear loads P and Q as seen in Fig. (A1).

A coordinate system (x, y) is located at the middle of the crack and the concentrated loads are applied at $(t, 0)$. Since the loads P and Q as appearing in that figure result in negative normal and shear stresses respectively adjacent to the point of application, they will be assigned negative values $P < 0$, $Q < 0$.

The stress fields σ_{xx}, σ_{yy}, σ_{xy} in generalized plane stress can be found from a pair of complex potential functions ϕ and Ω as follows (see for example Erdogan, 1962).

$$\sigma_{xx}+\sigma_{yy} = 4\mathrm{Re}\,[\phi(z)]$$
$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2[(\bar{z}-z)\phi'(z)-\phi(z)+\bar{\Omega}(z)] \tag{A1}$$

where $\bar{\Omega}(z)$ in the above equation is defined as

$$\bar{\Omega}(z) = \overline{\Omega(\bar{z})} \tag{A2}$$

with the functions ϕ, Ω and ϕ' are being given by

$$\phi(z) = -\frac{P-iQ}{2\pi(z-t)}\left(\frac{\ell^2-t^2}{z^2-\ell^2}\right)^{1/2}, \quad \Omega(z) = \phi(z) \tag{A3}$$

$$\phi'(z) = -\frac{P-iQ}{2\pi}(\ell^2-t^2)^{1/2}\frac{\ell^2+zt-2z^2}{(z-t)^2(z^2-\ell^2)^{1/2}}. \tag{A4}$$

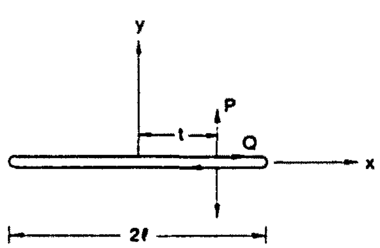

Fig. A1. Geometry of a solitary crack in an infinite medium loaded by concentrated unit loads.

The above equations provide a framework for the derivation of the appropriate stress functions for the case of a single crack loaded in mode I or mode II by a given distribution of normal and shear loadings, respectively. In the implementation of the crack interaction scheme which has been formulated in this paper two kinds of distributions will be needed: (a) power-type loadings (uniform, linear, quadratic, cubic, and quartic, etc.), (b) a piecewise constant loading in shear. Stress functions for the former category can be derived by following a procedure of contour integration. The stress function for the second type of loading can be obtained after a somewhat cumbersome integration procedure leading however to a rather simple result.

(A1) *Stress functions for power-type loadings*

Results will be given for the following loading cases:

$$p_0(t) = s_0(t) = -1 \quad p_1(t) = s_1(t) = -\left(\frac{t}{\ell}\right) \quad p_2(t) = s_2(t) = -\left(\frac{t}{\ell}\right)^2$$

$$p_3(t) = s_3(t) = -\left(\frac{t}{\ell}\right)^3 \quad p_4(t) = s_4(t) = -\left(\frac{t}{\ell}\right)^4. \tag{A5}$$

For the case of normal loading $p(t) = -f(t), f(t) > 0$, for example, the corresponding $\phi(z)$ is given by integrating the expression (A3) as follows:

$$\phi(z) = \frac{1}{2\pi(z^2-\ell^2)^{1/2}} \int_{-\ell}^{+\ell} \frac{[f(t)]}{(z-t)} (\ell^2-t^2)^{1/2}\, \mathrm{d}t. \tag{A6}$$

If $f(t)$ is a polynomial as is in the case of (A5), the integration of the definite integral (A6) can be carried out by a method described in Muskhelishvili (1953), p. 455. We first write (A6) in the form

$$\phi(z) = \frac{i}{2\pi(z^2-\ell^2)^{1/2}} \mathrm{I}(z) \tag{A7}$$

$$\mathrm{I}(z) = \int_{-\ell}^{+\ell} f(t) \frac{(t^2-\ell^2)^{1/2}}{(t-z)} \mathrm{d}t. \tag{A8}$$

For large t let,

$$(t^2-\ell^2)^{1/2} = t - \frac{\ell^2}{2t} - \frac{1}{8}\frac{\ell^4}{t^3} + \cdots$$

$$f(t)(t^2-\ell^2)^{1/2} = \alpha_q t^q + \alpha_{q-1} t^{q-1} + \cdots \alpha_0 + \alpha_{-1}\frac{1}{t} + \alpha_{-2}\frac{1}{t^2} + \cdots \tag{A9}$$

where q and α_i are constants. It can now be proven (see Muskhelishvili, 1953) that

$$\mathrm{I}(z) = \pi i[f(z)(z^2-\ell^2)^{1/2} - \alpha_q z - \cdots - \alpha_0]. \tag{A10}$$

For example in the case of $f(t) = 1$, $\mathrm{I}(z)$ is given

$$\mathrm{I}(z) = \pi i[(z^2-\ell^2)^{1/2} - z] \tag{A11}$$

and in the case of $f(t) = t/\ell$, we have

$$\mathrm{I}(z) = \frac{\pi i}{\ell}\left[(z^2-\ell^2)^{1/2} z - z^2 + \frac{\ell^2}{2}\right]. \tag{A12}$$

Let us now denote by $\phi_n^{(j)}, \Omega_n^{(j)}$ the potential functions corresponding to normal loading of type j as given in equation (A5), ($j = 0$, uniform $j = 1$, linear etc.). The potential functions corresponding to shear loadings will similarly be denoted by $\phi_s^{(j)}, \Omega_s^{(j)}$. The highest order base function $(-L_n(\xi))$ in this paper was with $n = 4$. The corresponding potentials functions could be derived by means of

$$\phi_n^{(0)} = \Omega_n^{(0)} = [2(z^2-\ell^2)^{1/2}]^{-1}[z-(z^2-\ell^2)^{1/2}]$$
$$\phi_n^{(1)} = \Omega_n^{(1)} = [2\ell(z^2-\ell^2)^{1/2}]^{-1}[z^2-z(z^2-\ell^2)^{1/2}-(\ell^2/2)]$$
$$\phi_n^{(2)} = \Omega_n^{(2)} = [2\ell^2(z^2-\ell^2)^{1/2}]^{-1}[z^3-z^2(z^2-\ell^2)^{1/2}-(z\ell^2/2)]$$
$$\phi_n^{(3)} = \Omega_n^{(3)} = [2\ell^3(z^2-\ell^2)^{1/2}]^{-1}[z^4-z^3(z^2-\ell^2)^{1/2}-(\ell^2 z^2/2)-(\ell^4/8)]$$
$$\phi_n^{(4)} = \Omega_n^{(4)} = [2\ell^4(z^2-\ell^2)^{1/2}]^{-1}[z^5-z^4(z^2-\ell^2)^{1/2}-(\ell^2 z^3/2)-(\ell^4 z/8)] \tag{A13}$$

and

$$\phi_s^{(j)} = \Omega_s^{(j)} = -i\phi_n^{(j)} \tag{A14}$$

(A.2) *Piecewise constant shear loading*

We consider here a shear loading as follows:

$$s^*(t) = \begin{cases} -1 & 0 \leqslant t \leqslant \ell \\ +1 & -\ell \leqslant t \leqslant 0. \end{cases} \tag{A15}$$

Again by integration of equation (3) the potential ϕ_s^* is given by

$$\phi_s^*(z) = \frac{i}{2\pi(z^2-\ell^2)^{1/2}}\left[\int_{-\ell}^{0} \frac{(\ell^2-t^2)^{1/2}}{(t-z)}\,\mathrm{d}t - \int_{0}^{+\ell} \frac{(\ell^2-t^2)^{1/2}}{(t-z)}\,\mathrm{d}t\right]. \tag{A16}$$

The indefinite integral

$$J(z,t) = \int \frac{(\ell^2-t^2)^{1/2}}{(t-z)}\,\mathrm{d}t \tag{A17}$$

can be evaluated by transforming it first into $t' = t-z$, which results in

$$J(z,t') = \int \frac{[-t'^2-2zt'-(z^2-\ell^2)]^{1/2}}{t'}\,\mathrm{d}t' \tag{A18}$$

and then using the formula (Bois, 1961)

$$\int \frac{(ax^2+x+c)^{1/2}}{x}\,\mathrm{d}x = (ax^2+bx+c)^{1/2} + \frac{b}{2}\frac{1}{\sqrt{a}}\ln\,[b+2ax+2\sqrt{a(ax^2+bx+c)}] + \sqrt{c}\,\ln\left[\frac{2c+bx-2\sqrt{c(ax^2+bx+c)}}{x}\right]. \tag{A19}$$

Identifying $a = -1$, $b = -2z$, $c = -(z^2-\ell^2)$ finally provides

$$J(z,t) = (\ell^2-t^2)^{1/2} + zi\,\ln\,[-2t+2i\sqrt{\ell^2-t^2}] + i(z^2-\ell^2)^{1/2}\ln\left(\frac{2\ell^2-2zt-2i\sqrt{(z^2-\ell^2)(\ell^2-t^2)}}{(t-z)}\right) \tag{A20}$$

leading to

$$\phi^*(z) = -\frac{i}{2\pi\sqrt{z^2-\ell^2}}\left\{2\ell + i(z^2-\ell^2)^{1/2}\left[2\,\ln\left(\frac{\ell-i\sqrt{z^2-\ell^2}}{-z}\right) - i\pi\right]\right\} \tag{A21}$$

with $\Omega^*(z) = \phi^*(z)$.

It is of interest to note that as $z \to 0$, $\phi^*(z)$ exhibits a logarithmic singularity resulting in a singularity of the same kind in the stresses.

The influence functions $f_{\beta\alpha}^{(n)}$, $g_{\beta\alpha}^{(n)}$, $h_{\beta\alpha}^{(n)}$, $q_{\beta\alpha}^{(n)}$ can be obtained through the use of (A13), (A21) and (A1).